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NAVAL SURFACE WARFARE CENTER**

Dahlgren, Virginia 22448-5100



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**ANALYSIS OF THE PROBABILITY DENSITY  
FUNCTION OF THE MONOPULSE RATIO RADAR  
SIGNAL**

**BY WINSTON C. CHOW**

**WARFARE ANALYSIS DEPARTMENT**

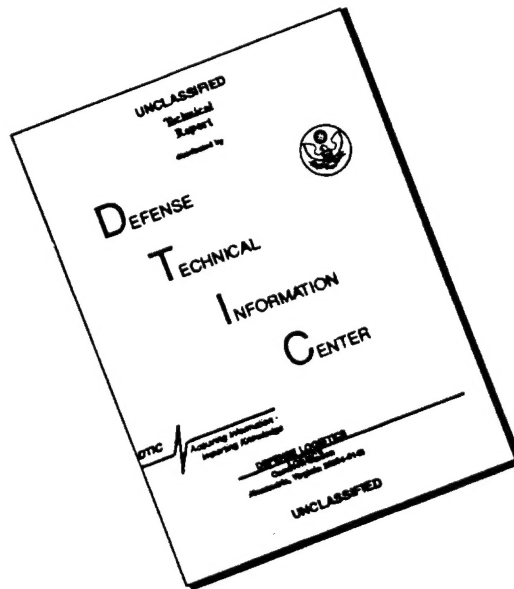
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Winston C. Chow

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**Commander  
Naval Surface Warfare Center  
Dahlgren Division (Code A52)  
17320 Dahlgren Road  
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Monopulse radars are often used by the U.S. Navy Department to track airborne targets such as missiles attacking ships. In order to effectively make use of the target position data provided by the radar, an estimate of the radar output accuracy must be known. This report describes portions of research performed at Naval Surface Warfare Center, Dahlgren Division (NSWCDD) to develop statistical models of monopulse ratio behavior to improve accuracy analysis of the monopulse radar.

The products of this work include a formula for the distribution of the real valued part of the radar signal. Also developed were two tools to compare this distribution with a normal distribution. With these formulas, we showed that, given certain assumptions, this distribution can be approximated by a normal distribution. Finally, this work showed the validity of certain pre-existing formulas developed at NSWCDD. These results are useful in developing algorithms for assessing the accuracy of target location estimates by monopulse radars.

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## FOREWORD

Monopulse radars are often used by the U.S. Navy Department to track low elevation airborne targets such as missiles attacking ships. In order to effectively make use of the target position data provided by the radar, an estimate of the radar output accuracy must be known. This accuracy information is necessary in order to judge how much distance and at what angle the actual target can be located away from the radar estimated location. Such information is necessary for defensive actions against the target. Also, radar accuracy information is needed in order to make efficient use of the radar equipment itself since the amount of time the radar requires in tracking the specific position of a target increases with the desired accuracy.

This report describes portions of research performed at the Naval Surface Warfare Center, Dahlgren Division (NSWCDD) to develop statistical models of monopulse ratio behavior to improve accuracy analysis of the monopulse radar. The products of this work are mainly new formulas and confirmations of existing formulas. These results are useful in developing algorithms for assessing the accuracy of target location estimates by monopulse radars.

The author would like to thank Gordon W. Groves and William D. Blair, both of NSWCDD, Code B32, for their suggestions for investigating the problems discussed in this report. The work described herein was performed to support their independent research project, which was funded by the NSWCDD/Dahlgren In-house Laboratory Independent Research (ILIR) Program. The author would also like to thank Isaac Sofair of Code K41 at NSWCDD for helping to examine some of the problems encountered during this research.

This report has been reviewed by Soon Leong (Code F42), Larry Beuglass (Acting Head, Systems Effectiveness Branch), and Alan R. Glazman (Head, Cost and Effectiveness Analysis Division). Their efforts are greatly appreciated.

Approved by:



C. A. KALIVRETENOS, Head  
Warfare Analysis Department

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## CHAPTER 1

### INTRODUCTION

Monopulse radars are used by the U.S. Navy Department for tracking low elevation airborne targets such as missiles attacking Naval ships. One such missile in this category is the Antiship Cruise Missile. In order to effectively make use of the target position data provided by the radar, an estimate of the radar output accuracy must be known. This accuracy information is necessary in order to judge how much distance and angle the actual target can be located away from the radar estimated location. Such information is necessary for defensive actions against the target. Also, radar accuracy information is needed in order to make efficient use of the radar equipment itself since the amount of time the radar requires in tracking the specific position of a target increases with the desired accuracy.

As with any signal processing instrument, random disturbances, commonly known as random noise, are generated during its use. These disturbances occur from both sources external to the radar as well as from the instrument itself. The random noises in the radar signal form errors in the radar reading that constitute a significant source of inaccuracy in target position estimation. Analyzing the nature of this random noise is useful in minimizing the impact of such errors and making efficient use of radar resources.

Another reason of why a better understanding of random disturbances is useful is related to how radar tracking is performed. Estimating algorithms such as the Kalman filter are used in the tracking software in order to estimate the position, velocity, and/or acceleration of the target. One of the inputs of such estimators is the measurement error covariance. In the past, there have been no adequate approximations of this quantity. One other application of the study of random disturbances is a better understanding of how and (to what extent) the errors caused by such noise propagate through the radar system, causing inaccuracy in the final reading.

Due to the random nature of this noise, deterministic mathematical methods do not form adequate models of radar signals. Hence, radar signals needs to be studied from a statistical and probabilistic point of view. There has been research work that performed on the statistical nature of monopulse ratio radar signals. Some of this work was documented in References 1, 2, and 3. The additional new research shown in this present document supplements and extends these investigations. The products of this work are mainly in the forms of new formulas as well as confirmations of existing formulas. These results are useful in developing algorithms for assessing the accuracy of target location estimates by monopulse radars.

In analyzing the statistical nature of a monopulse radar, any information on the nature of the joint probability density function of the complex valued monopulse ratio is important for specifying the statistical parameters. This joint probability density function for various cases has been derived in Reference 1, the distribution of the real part of the monopulse ratio was then still unknown. Yet, the probability density function of the real part of the complex monopulse ratio along with any other properties that can be derived to further describe this probability density function is useful in determining the errors of the angular displacement of the target being tracked by the radar. The research described in this report helped to characterize the real part of

the monopulse ratio. In this report, the probability density of the real part of the monopulse ratio for the zero mean and partially correlated case is derived based upon the joint density described in Reference 1. In addition, two methods of analyzing this probability density function to help determine its closeness to a normal (Gaussian) distribution are described. Furthermore, a comparative study is performed that compares some of the results given by Seifer (References 2 and 3) to the results reported by Groves and Blair (Reference 1). In order to enhance clarity, an explanation of the equivalence of a complex valued random variable to a bivariate random vector is given in the last section of this report. The derivation of the variance of the real part of the probability density function for the zero mean and partially correlated case is described in Appendix A. This variance formula is needed in one of the methods for analyzing the closeness of the probability density to the normal density.



## CHAPTER 2

## PROBABILITY DENSITY OF THE REAL PART OF THE MONOPULSE RATIO

Although the bivariate density of the monopulse ratio with partial correlations is known (Reference 1), the density of the real part of the monopulse ratio was not yet derived. In this section, the probability density function of the real part of the monopulse ratio is derived as the marginal density of the joint probability density function of the real and imaginary portions of the monopulse ratio. The magnitude of the signal is assumed to exceed a given threshold level in order to be considered. This is further explained in the next paragraph.

In a monopulse radar, the signal consists of four beams, which is modeled as four random variables,  $X$ ,  $Y$ ,  $U$ , and  $V$ . Each of these random variables is assumed to be normally distributed. Furthermore, these four forms of the signals are transformed into the sum and difference channels. Let  $S=U+iV$  be the sum channel of the monopulse ratio and  $D=X+iY$  be the difference channel. Let  $P+iQ=R=D/S$  be the monopulse ratio and  $\rho$  be the correlation between  $x$  and  $u$  (the real parts), which is assumed to be the same as that of  $y$  and  $v$  (the imaginary parts). Furthermore, let  $\sigma_X$  and  $\sigma_U$  represent the standard deviations of  $x$  and  $u$ , respectively, and  $\gamma=\sigma_X/\sigma_U$ . The threshold assumption, mentioned in the previous paragraph, is that the magnitude of the sum channel must exceed a given positive constant  $R_0$  in order to be considered a true signal. In other words, the assumption states that in order to be recognized as a legitimate signal, the signal must be such that  $\sqrt{(U^2+V^2)} > R_0 > 0$ .

Suppose that the monopulse ratio is zero mean and has the following covariance matrix:

$$\Sigma = \sigma_u^2 \begin{bmatrix} \gamma^2 & 0 & \rho\gamma & 0 \\ 0 & \gamma^2 & 0 & \rho\gamma \\ \rho\gamma & 0 & 1 & 0 \\ 0 & \rho\gamma & 0 & 1 \end{bmatrix} \quad (2.1)$$

Then, the joint probability density function of the real and imaginary parts of the monopulse ratio is shown in Equation (7.12) of Reference 1 to be as follows:

$$P_{P,Q}(p, q) = \frac{\gamma^2 (1 - \rho^2)}{\pi} \left[ \frac{h^2 (p^2 + q^2 + \gamma^2 - 2\rho\gamma p) + 1}{(p^2 + q^2 + \gamma^2 - 2\rho\gamma p)^2} \right] e^{-h^2 (p^2 + q^2 + \gamma^2 - 2\rho\gamma p)} \quad (2.2)$$

where  $h$  is a "normalized" threshold, defined as

$$h = \frac{R_0}{\sigma_x \sqrt{2(1-\rho)}} \quad (2.3)$$

and  $R_0$  is the actual threshold level. For the sake of completeness, one may note that a general probability density function for  $P$  and  $Q$  with any given correlation of the noises and without the threshold restriction was derived and given in Equation (6.12) of Reference 1. Definitions of various variables in this equation are given by Equations (6.2), (6.6), and (6.13) of Reference 1. This formula for the general probability density function for  $P$  and  $Q$  was later empirically confirmed. The empirical confirmation was done by programming the formula using the MATLAB computer language and numerically integrating the formula over all  $p$  and  $q$ . (The numerical result of this integration was one. See Appendix B for more details.)

The probability density of the real part,  $P$ , of the monopulse ratio can be derived as the integral over the real line of the joint density with respect to  $q$ . Hence,

$$P_P(p) = \quad (2.4)$$

$$\frac{\gamma^2(1-\rho^2)}{\pi} \int_{-\infty}^{\infty} \left[ \frac{h^2(p^2 + q^2 + \gamma^2 - 2\rho\gamma p) + 1}{(p^2 + q^2 + \gamma^2 - 2\rho\gamma p)^2} \right] \exp[-h^2(p^2 + q^2 + \gamma^2 - 2\rho\gamma p)] dq$$

Let  $c_1 = p^2 + \gamma^2 - 2\rho\gamma p$  and  $c_2 = p^2 + \rho^2\gamma^2 - 2\rho\gamma p$ . Since by definition of correlation, the magnitude of  $\rho$  is no greater than one and  $p^2 + \gamma^2 \geq 2\rho\gamma p$ , one can conclude that  $c_1 = p^2 + \gamma^2 - 2\rho\gamma p \geq 0$ . Therefore,  $c_1$ , being a non-negative number, can be represented as  $c_1 = c^2$  for some quantity  $c$ . Using these symbols,

$$P_P(p) = \quad (2.5)$$

$$\begin{aligned} & \frac{\gamma^2(1-\rho^2)}{\pi} \int_{-\infty}^{\infty} \left[ \frac{h^2(q^2 + c^2) + 1}{(q^2 + c^2)^2} \right] \exp[-h^2(q^2 + c^2)] dq \\ &= \frac{\gamma^2(1-\rho^2)}{\pi} \exp(-h^2 c^2) \left[ \int_{-\infty}^{\infty} \frac{h^2}{(q^2 + c^2)^2} e^{-h^2 q^2} dq + \int_{-\infty}^{\infty} \frac{1}{(q^2 + c^2)^2} e^{-h^2 q^2} dq \right] \end{aligned}$$

One can see that this probability density contains the sum of two integrals. Consider the first integral. Looking at Equation (3.466) of Reference 4,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{h^2}{(q^2 + c^2)} e^{-h^2 q^2} dq &= 2h^2 \int_0^{\infty} \frac{e^{-h^2 q^2}}{(q^2 + c^2)} dq \\ &= 2h^2 [1 - \phi(ch)] \frac{\pi}{2c} e^{h^2 c^2} \end{aligned} \quad (2.6)$$

where

$$\phi(ch) = \frac{2}{\sqrt{\pi}} \int_0^{ch} e^{-t^2} dt \Rightarrow \quad (2.7)$$

$$\int_{-\infty}^{\infty} \frac{h^2}{(q^2 + c^2)} e^{-h^2 q^2} dq = \frac{\pi h^2}{c} e^{h^2 c^2} [1 - \phi(ch)]$$

Now consider the second integral in the formula for  $P_p(p)$ . Note that

$$\begin{aligned} \frac{d}{dc} \int_{-\infty}^{\infty} \frac{h^2}{(q^2 + c^2)} e^{-h^2 q^2} dq &= -2h^2 c \int_{-\infty}^{\infty} \frac{1}{(q^2 + c^2)^2} e^{-h^2 q^2} dq \Rightarrow \\ \int_{-\infty}^{\infty} \frac{1}{(q^2 + c^2)^2} e^{-h^2 q^2} dq &= \left( \frac{-1}{2h^2 c} \right) \frac{d}{dc} \left( \int_{-\infty}^{\infty} \frac{h^2}{(q^2 + c^2)} e^{-h^2 q^2} dq \right) \\ &= \left( \frac{-1}{2h^2 c} \right) \frac{d}{dc} \left[ \frac{\pi h^2}{c} e^{h^2 c^2} (1 - \phi(ch)) \right] \\ &= \left( \frac{-\pi}{2c} \right) \frac{d}{dc} \left[ \frac{e^{h^2 c^2}}{c} (1 - \phi(ch)) \right] \end{aligned} \quad (2.8)$$

Taking the derivative of the first factor,

$$\begin{aligned}\frac{d}{dc} \left( \frac{\exp(h^2 c^2)}{c} \right) &= \exp(h^2 c^2) \left( 2h^2 - \frac{1}{c^2} \right) \Rightarrow \\ \frac{d}{dc} \left[ \frac{\exp(h^2 c^2)}{c} (1 - \phi(ch)) \right] &= \exp(h^2 c^2) \left( 2h^2 - \frac{1}{c^2} \right) (1 - \phi(ch)) - \\ &\quad \frac{\exp(h^2 c^2)}{c} \frac{d}{dc} \phi(ch)\end{aligned}\tag{2.9}$$

With the derivations just given, one can conclude that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-h^2 q^2}}{(q^2 + c^2)^2} dq &= \frac{-\pi}{2c} e^{h^2 c^2} \left[ \left( 2h^2 - \frac{1}{c^2} \right) (1 - \phi(ch)) - \frac{1}{c} \frac{d}{dc} \phi(ch) \right] \\ &= \frac{-\pi}{2c^2} e^{h^2 c^2} \left[ \left( \frac{2h^2 c^2 - 1}{c} \right) (1 - \phi(ch)) - \frac{d}{dc} \phi(ch) \right]\end{aligned}$$

where (2.10)

$$\frac{d}{dc} \phi(ch) = \frac{d}{dc} \left( \frac{2}{\sqrt{\pi}} \int_0^{ch} e^{-t^2} dt \right) = \frac{2h}{\sqrt{\pi}} e^{-c^2 h^2}$$

Note that the formula for the derivative of  $\phi(ch)$  was derived by using the Leibniz theorem. Now putting the two integrals together,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{h^2}{(q^2 + c^2)} e^{-h^2 q^2} dq + \int_{-\infty}^{\infty} \frac{1}{(q^2 + c^2)^2} e^{-h^2 q^2} dq \\ = \frac{\pi e^{h^2 c^2}}{c} \{ h^2 [1 - \phi(ch)] - \frac{1}{2c} \left[ \left( \frac{2h^2 c^2 - 1}{c} \right) (1 - \phi(ch)) - \frac{d}{dc} \phi(ch) \right] \}\end{aligned}\tag{2.11}$$

Now recalling the equation for  $P_P(p)$  given by Equation (2.5), the following must hold true:

$$\begin{aligned}
 P_P(p) &= \frac{\gamma^2 (1 - \rho^2)}{c} e^{h^2 c^2} e^{-h^2 c_2} \{h^2 [1 - \phi(ch)] - \\
 &\quad \frac{1}{2c} [(\frac{2h^2 c^2 - 1}{c})(1 - \phi(ch)) - \frac{d}{dc} \phi(ch)]\} \\
 &= \frac{\gamma^2 (1 - \rho^2)}{c} e^{h^2 c^2} e^{-h^2 c_2} \{h^2 [1 - \phi(ch)] - \frac{1}{2c} [(\frac{2h^2 c^2 - 1}{c})(1 - \phi(ch))]\} \\
 &\quad + (\frac{1}{2c})(\frac{2h}{\sqrt{\pi}}) e^{-c^2 h^2} \}
 \end{aligned} \tag{2.12}$$

$$\begin{aligned}
 \therefore P_P(p) &= \frac{\gamma^2 (1 - \rho^2)}{c} e^{h^2 c^2} e^{-h^2 c_2} \{h^2 [1 - \phi(ch)] - \frac{1}{2c} [(\frac{2h^2 c^2 - 1}{c})(1 - \phi(ch))]\} \\
 &\quad + \frac{\gamma^2 (1 - \rho^2)}{c^2 \sqrt{\pi}} h e^{-h^2 c_2}
 \end{aligned} \tag{2.13}$$

with  $c_2 = p^2 + \rho^2 \gamma^2 - 2\rho\gamma p$  and  $c^2 = c_1$  and  $c_1 = p^2 + \gamma^2 - 2\rho\gamma p$ , as were previously defined. So the formula for the probability density function of the real portion,  $P$ , of the monopulse ratio has now been derived.

This formula will now be modified to a form that can be more easily programmed on a computer for use in monopulse ratio radar signal processing. The given formula for  $P_P(p)$  implies that

$$\begin{aligned}
 P_P(p) &= \frac{\gamma^2 (1 - \rho^2)}{c} e^{h^2 \gamma^2 (1 - \rho^2)} \{[h^2 - (\frac{2h^2 c^2 - 1}{2c^2})][1 - \phi(ch)]\} \\
 &\quad + \frac{\gamma^2 (1 - \rho^2)}{c^2 \sqrt{\pi}} h e^{-h^2 c_2}
 \end{aligned} \tag{2.14}$$

This implies that

$$P_P(p) = \frac{\gamma^2 (1 - \rho^2)}{2c^3} [1 - \phi(ch)] e^{h^2 \gamma^2 (1 - \rho^2)} + \frac{\gamma^2 (1 - \rho^2)}{c^2 \sqrt{\pi}} h e^{-h^2 c^2} \quad (2.15)$$

$$\therefore P_P(p) = \frac{\gamma^2 (1 - \rho^2)}{2c^3} \left\{ [1 - \phi(ch)] e^{h^2 \gamma^2 (1 - \rho^2)} + \frac{2ch}{\sqrt{\pi}} e^{-h^2 c^2} \right\} \quad (2.16)$$

Notice that the probability density function of  $P$  is obviously not normally distributed. In the next two sections, formulas will be derived that can be used to provide some form of measuring how close the real part of the monopulse ratio, whose probability density function was just derived, is to a Gaussian density function. The first method exploits the fact that any Gaussian distribution has a kurtosis value of 3, and the second method uses a stochastic distance measure.

### CHAPTER 3

#### FOURTH MOMENT ABOUT THE MEAN AND KURTOSIS OF THE REAL PART OF THE MONOPULSE RATIO

One method of giving a measure of the closeness between the real part,  $P$ , of the monopulse ratio and a normally distributed random variable uses the fact that the kurtosis of any Gaussian (that is, normal) distribution has a value of 3. Hence, finding the kurtosis of  $P$  and comparing it with 3 is one method of comparison of the closeness of  $P$  to a Gaussian distribution. Certainly, there can be distributions other than Gaussian that also have a kurtosis of 3, but this is still one form of comparison between  $P$  and a normally distributed random variable. Furthermore, deriving the formulas for the fourth moment of  $P$  along with its kurtosis may themselves be useful information concerning the statistical nature of monopulse ratio radars.

Recall that kurtosis is defined as in the following.

Definition: Let  $X$  be a random variable. The kurtosis of  $X$  is given as

$$\text{Kurtosis}(X) \equiv \frac{E[(X - E[X])^4]}{\sigma_X^4} \quad (3.1)$$

In order to simplify notations, let  $t = P - E[P]$ . Also, let  $E_1$  be defined as the following function for any given real number  $y$ :

$$E_1(y) \equiv \int_y^{\infty} \frac{e^{-u}}{u} du \quad (3.2)$$

Also, let  $\gamma$  and  $\rho$  be as previously defined. One may note the fact that according to Reference 1 (on the top of page 7-3),

$$E[P] = \rho\gamma \quad (3.3)$$

The fourth moment of  $P$  about the mean,  $\rho\gamma$ , will now be derived. With the notations that were just given, the fourth moment about the mean can be denoted by  $E[t^4]$ . In finding the fourth moment, a transformation from the Cartesian coordinates to the polar coordinates will be

made. Hence, let  $t=P-E[P]=R\cos\theta$ . From Equation (7.16) of Reference 1 the variance (second moment about the mean) of  $P$  is given by the following equation:

$$E[t^2] = \frac{\gamma^2(1-\rho^2)}{\pi} \int_0^{2\pi} \cos^2 \theta d\theta \int_0^\infty R^3 dR \left[ \frac{h^2[R^2 + \gamma^2(1-\rho^2)] + 1}{[R^2 + \gamma^2(1-\rho^2)]^2} \right] \exp[-h^2 R^2] \quad (3.4)$$

Writing the new variable  $t$  in the form of  $t=R\cos\theta$ , the above formula implies that  $E[t^4]$  can be represented as follows:

$$E[t^4] = \frac{\gamma^2(1-\rho^2)}{\pi} \int_0^{2\pi} \cos^4 \theta d\theta \int_0^\infty R^5 dR \left[ \frac{h^2[R^2 + \gamma^2(1-\rho^2)] + 1}{[R^2 + \gamma^2(1-\rho^2)]^2} \right] \exp[-h^2 R^2] \quad (3.5)$$

Looking at the first integral in this equation, the following relationships must be true.

$$\int_0^{2\pi} \cos^4 \theta d\theta = \left[ \frac{3}{8} \theta + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} \right]_0^{2\pi} = \frac{3}{8} (2\pi) = \frac{3}{4} \pi \Rightarrow \quad (3.6)$$

$$E[t^4] = \frac{3}{4} \gamma^2(1-\rho^2) \int_0^\infty R^5 dR \left[ \frac{h^2[R^2 + \gamma^2(1-\rho^2)] + 1}{[R^2 + \gamma^2(1-\rho^2)]^2} \right] \exp(-h^2 R^2) \quad (3.7)$$

In performing this integral, another change of variables will now be made. Let  $u=R^2$ . This means that  $du=2RdR$  and  $RdR=du/2$ . Substituting these values,

$$E[t^4] = \frac{3}{8} \gamma^2(1-\rho^2) \int_0^\infty u^2 du \left[ \frac{h^2[u + \gamma^2(1-\rho^2)] + 1}{[u + \gamma^2(1-\rho^2)]^2} \right] \exp(-h^2 u) \quad (3.8)$$



Now let  $\alpha = u + \gamma^2(1 - \rho^2)$ . This implies that  $d\alpha = du$  and  $u = \alpha - \gamma^2(1 - \rho^2)$ . Substituting these values,

$$\begin{aligned}
 E[t^4] &= \\
 &= \frac{3}{8} \gamma^2 (1 - \rho^2) \int_{\gamma^2(1-\rho^2)}^{\infty} [\alpha - \gamma^2(1 - \rho^2)]^2 d\alpha \left( \frac{h^2 \alpha + 1}{\alpha^2} \right) \exp(-h^2 \alpha) \exp(h^2 \gamma^2 (1 - \rho^2)) \\
 &= \frac{3}{8} \gamma^2 (1 - \rho^2) e^{h^2 \gamma^2 (1 - \rho^2)} \int_{\gamma^2(1-\rho^2)}^{\infty} [\alpha - \gamma^2(1 - \rho^2)]^2 d\alpha \left( \frac{h^2 \alpha + 1}{\alpha^2} \right) e^{-h^2 \alpha} = \\
 &= \frac{3}{8} \gamma^2 (1 - \rho^2) e^{h^2 \gamma^2 (1 - \rho^2)} \int_{\gamma^2(1-\rho^2)}^{\infty} [\alpha^2 - 2\alpha \gamma^2 (1 - \rho^2) + \gamma^4 (1 - \rho^2)^2] d\alpha * \\
 &\quad \left( \frac{h^2 \alpha + 1}{\alpha^2} \right) e^{-h^2 \alpha}
 \end{aligned} \tag{3.9}$$

In order to further simplify formulas, let

$$K = (3/8) \gamma^2 (1 - \rho^2) \exp(h^2 \gamma^2 (1 - \rho^2)) \tag{3.10}$$

Thus,

$$\begin{aligned}
 E[t^4] &= K \left\{ \int_{\gamma^2(1-\rho^2)}^{\infty} (h^2 \alpha + 1) e^{-h^2 \alpha} d\alpha - 2 \gamma^2 (1 - \rho^2) \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{h^2 \alpha + 1}{\alpha} \right) e^{-h^2 \alpha} d\alpha \right. \\
 &\quad \left. + \gamma^4 (1 - \rho^2)^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{h^2 \alpha + 1}{\alpha^2} \right) e^{-h^2 \alpha} d\alpha \right\}
 \end{aligned} \tag{3.11}$$

$$\begin{aligned}
 \therefore E[t^4] &= K \left\{ \int_{\gamma^2(1-\rho^2)}^{\infty} h^2 \alpha e^{-h^2 \alpha} d\alpha + \int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2 \alpha} d\alpha - 2 \gamma^2 (1 - \rho^2) h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2 \alpha} d\alpha \right. \\
 &\quad - 2 \gamma^2 (1 - \rho^2) \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{1}{\alpha} \right) e^{-h^2 \alpha} d\alpha + \gamma^4 (1 - \rho^2)^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{h^2}{\alpha} \right) e^{-h^2 \alpha} d\alpha \\
 &\quad \left. + \gamma^4 (1 - \rho^2)^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{1}{\alpha^2} \right) e^{-h^2 \alpha} d\alpha \right\}
 \end{aligned} \tag{3.12}$$

$$\begin{aligned}
\therefore E[t^4] &= K \left\{ \int_{\gamma^2(1-\rho^2)}^{\infty} h^2 \alpha e^{-h^2 \alpha} d\alpha + \int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2 \alpha} d\alpha - 2\gamma^2(1-\rho^2)h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2 \alpha} d\alpha \right. \\
&\quad - 2\gamma^2(1-\rho^2) \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha}\right) e^{-h^2 \alpha} d\alpha + \gamma^4(1-\rho^2)^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{h^2}{\alpha}\right) e^{-h^2 \alpha} d\alpha \\
&\quad \left. + \gamma^4(1-\rho^2)^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha^2}\right) e^{-h^2 \alpha} d\alpha \right\} \\
&= K \left\{ h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \alpha e^{-h^2 \alpha} d\alpha + [1 - 2\gamma^2(1-\rho^2)h^2] \int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2 \alpha} d\alpha + \right. \\
&\quad \left. \gamma^2(1-\rho^2)[\gamma^2(1-\rho^2)h^2 - 2] \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{e^{-h^2 \alpha}}{\alpha}\right) d\alpha + \gamma^4(1-\rho^2)^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{e^{-h^2 \alpha}}{\alpha^2}\right) d\alpha \right\},
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\int_{\gamma^2(1-\rho^2)}^{\infty} \alpha e^{-h^2 \alpha} d\alpha &= \frac{e^{-h^2 \alpha}}{h^4} (-h^2 \alpha - 1) \Big|_{\gamma^2(1-\rho^2)}^{\infty} \\
&= [h^2 \gamma^2(1-\rho^2) + 1] \frac{e^{-h^2 \gamma^2(1-\rho^2)}}{h^4}
\end{aligned} \tag{3.13}$$

$$\text{and } \int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2 \alpha} d\alpha = \frac{e^{-h^2 \alpha}}{-h^2} \Big|_{\gamma^2(1-\rho^2)}^{\infty} = \frac{e^{-h^2 \gamma^2(1-\rho^2)}}{h^2} \tag{3.14}$$

Now consider the integral term,

$$\int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{e^{-h^2 \alpha}}{\alpha^2}\right) d\alpha \tag{3.15}$$

$$\text{Let } u = e^{-h^2 \alpha} \text{ and } dv = \alpha^{-2} d\alpha \Rightarrow du = -h^2 e^{-h^2 \alpha} \text{ and } v = -\alpha^{-1}$$

Substituting and then integrating,

$$\begin{aligned} \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-h^2\alpha}}{\alpha^2} \right) d\alpha &= -\alpha^{-1} e^{-h^2\alpha} \Big|_{\gamma^2(1-\rho^2)}^{\infty} - h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-h^2\alpha}}{\alpha} \right) d\alpha \\ &= \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-h^2\alpha}}{\alpha} \right) d\alpha \end{aligned} \quad (3.16)$$

Now let  $w=h^2\alpha$ . Hence,  $dw=h^2d\alpha$ , implying that  $d\alpha=dw/h^2$ . Then note that

$$\begin{aligned} \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-h^2\alpha}}{\alpha} \right) d\alpha &= \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - \\ h^2 \gamma^2(1-\rho^2) \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-w}}{w/h^2} \right) dw &= \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-w}}{w} \right) dw \\ &= \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 E_1[h^2\gamma^2(1-\rho^2)], \end{aligned} \quad (3.17)$$

where  $E_1$  is a previously defined function. This derivation shows that

$$\int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-h^2\alpha}}{\alpha^2} \right) d\alpha = \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 E_1[h^2\gamma^2(1-\rho^2)] \quad (3.18)$$

Now consider the integral term,  $\int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-h^2\alpha}}{\alpha} \right) d\alpha$ . Again letting  $w=h^2\alpha$  giving

$dw=h^2d\alpha$ , implying that  $d\alpha=dw/h^2$ . Using these terms,

$$\begin{aligned} \int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-h^2\alpha}}{\alpha} \right) d\alpha &= \int_{h^2\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-u}}{w/h^2} \right) \frac{dw}{h^2} \\ &= \int_{h^2\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-w}}{w} \right) dw = E_1 [h^2\gamma^2(1-\rho^2)] \Rightarrow \end{aligned} \quad (3.19)$$

$$\int_{\gamma^2(1-\rho^2)}^{\infty} \left( \frac{e^{-h^2\alpha}}{\alpha} \right) d\alpha = E_1 [h^2\gamma^2(1-\rho^2)] \quad (3.20)$$

Putting everything together,

$$\begin{aligned} E[t^4] &= \\ &Kh^2[h^2\gamma^2(1-\rho^2)+1] \frac{e^{-h^2\gamma^2(1-\rho^2)}}{h^4} + K[1-2h^2\gamma^2(1-\rho^2)] \frac{e^{-h^2\gamma^2(1-\rho^2)}}{h^2} + \\ &K\gamma^2(1-\rho^2)[h^2\gamma^2(1-\rho^2)-2]E_1[h^2\gamma^2(1-\rho^2)] + \end{aligned} \quad (3.21)$$

$$\begin{aligned} &K\gamma^4(1-\rho^2)^2 \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - K[\gamma^4(1-\rho^2)^2]h^2E_1[h^2\gamma^2(1-\rho^2)] \\ &= \gamma^2K(1-\rho^2)e^{-h^2\gamma^2(1-\rho^2)} + \frac{Ke^{-h^2\gamma^2(1-\rho^2)}}{h^2} + \frac{Ke^{-h^2\gamma^2(1-\rho^2)}}{h^2} - \\ &2\gamma^2K(1-\rho^2)e^{-h^2\gamma^2(1-\rho^2)} - 2\gamma^2K(1-\rho^2)E_1[h^2\gamma^2(1-\rho^2)] + \\ &\gamma^4K(1-\rho^2)^2 \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} \end{aligned} \quad (3.22)$$

$$= 2 \frac{K e^{-h^2 \gamma^2 (1-\rho^2)}}{h^2} - \gamma^2 K (1-\rho^2) e^{-h^2 \gamma^2 (1-\rho^2)} + \quad (3.23)$$

$$K \gamma^4 (1-\rho^2)^2 \frac{e^{-h^2 \gamma^2 (1-\rho^2)}}{\gamma^2 (1-\rho^2)} - 2 K \gamma^2 (1-\rho^2) E_1 [h^2 \gamma^2 (1-\rho^2)]$$

$$= 2 K \left\{ \frac{e^{-h^2 \gamma^2 (1-\rho^2)}}{h^2} - \gamma^2 (1-\rho^2) E_1 [h^2 \gamma^2 (1-\rho^2)] \right\} \quad (3.24)$$

Now recall K was previously defined to be  $K = (3/8) \gamma^2 (1-\rho^2) \exp(h^2 \gamma^2 (1-\rho^2))$ . Substituting this expression for K,

$$2 K \left\{ \frac{e^{-h^2 \gamma^2 (1-\rho^2)}}{h^2} - \gamma^2 (1-\rho^2) E_1 [h^2 \gamma^2 (1-\rho^2)] \right\} =$$

$$\frac{3}{4} \gamma^2 (1-\rho^2) e^{h^2 \gamma^2 (1-\rho^2)} \left\{ \frac{e^{-h^2 \gamma^2 (1-\rho^2)}}{h^2} - \gamma^2 (1-\rho^2) E_1 [h^2 \gamma^2 (1-\rho^2)] \right\} \quad (3.25)$$

$$= \left( \frac{3}{4} \right) \frac{\gamma^2 (1-\rho^2)}{h^2} - \left( \frac{3}{4} \right) \gamma^4 (1-\rho^2)^2 e^{h^2 \gamma^2 (1-\rho^2)} E_1 [h^2 \gamma^2 (1-\rho^2)]$$

$$\therefore E[t^4] = \quad (3.26)$$

$$\frac{3}{4} \gamma^2 (1-\rho^2) \left\{ \frac{1}{h^2} - \gamma^2 (1-\rho^2) E_1 [h^2 \gamma^2 (1-\rho^2)] e^{h^2 \gamma^2 (1-\rho^2)} \right\}$$

Hence, the equation for the fourth moment about the mean of P has finally been derived.

Here are some observations of this fourth moment formula. Notice that from definitions already given,

$$E_1 [h^2 \gamma^2 (1-\rho^2)] = \int_{h^2 \gamma^2 (1-\rho^2)}^{\infty} \frac{e^{-u}}{u} du, \quad \gamma = \frac{\sigma_x}{\sigma_u},$$

$$\text{and } \frac{1}{h^2} = \frac{2 \sigma_x^2 (1-\rho^2)}{R_0^2} \quad (3.27)$$

The last two of the above three expressions implies that

$$\frac{1}{h^2} = \frac{2\gamma^2(1-\rho^2)}{R_0^2} \sigma_u^2 \quad (3.28)$$

Using the formula for the fourth moment about the mean,  $E[t^4]$ , the kurtosis of P can now be derived, based upon the definition of kurtosis and the variance of P.

The variance of P is given in Equation (7.17) of Reference 1. However, there is a slight error in that formula since there should be no negative sign in the exponential expression. The corrected formula for the variance, using the t to represent  $P-E[P]$ , as before, is

$$E[t^2] = \frac{\gamma^2(1-\rho^2)}{2} \exp(h^2\gamma^2(1-\rho^2)) E_1[h^2\gamma^2(1-\rho^2)] \quad (3.29)$$

See Appendix A for a derivation of this corrected result. Using the definition for kurtosis,

$$\begin{aligned} \text{Kurtosis}(P) &\equiv \frac{E[(P - E[P])^4]}{\sigma^4} = \\ &\frac{(3/4)\gamma^2(1-\rho^2)\left[\frac{1}{h^2} - \gamma^2(1-\rho^2)E_1[h^2\gamma^2(1-\rho^2)]e^{h^2\gamma^2(1-\rho^2)}\right]}{(1/4)\gamma^4(1-\rho^2)^2 e^{2h^2\gamma^2(1-\rho^2)} E_1^2[h^2\gamma^2(1-\rho^2)]} \end{aligned} \quad (3.30)$$

$$\begin{aligned} &= \frac{3}{\gamma^2(1-\rho^2)} \left\{ \frac{e^{-2h^2\gamma^2(1-\rho^2)}}{h^2 E_1^2[h^2\gamma^2(1-\rho^2)]} - \frac{\gamma^2(1-\rho^2)}{e^{h^2\gamma^2(1-\rho^2)} E_1[h^2\gamma^2(1-\rho^2)]} \right\} \end{aligned} \quad (3.31)$$

$$\begin{aligned} &= \frac{3}{h^2\gamma^2(1-\rho^2)} \left\{ \frac{e^{-2h^2\gamma^2(1-\rho^2)}}{E_1^2[h^2\gamma^2(1-\rho^2)]} - \frac{h^2\gamma^2(1-\rho^2)e^{-h^2\gamma^2(1-\rho^2)}}{E_1[h^2\gamma^2(1-\rho^2)]} \right\} \\ &= \frac{3e^{-h^2\gamma^2(1-\rho^2)}}{h^2\gamma^2(1-\rho^2)} \left\{ \frac{e^{-h^2\gamma^2(1-\rho^2)}}{E_1^2[h^2\gamma^2(1-\rho^2)]} - \frac{h^2\gamma^2(1-\rho^2)}{E_1[h^2\gamma^2(1-\rho^2)]} \right\} \end{aligned} \quad (3.32)$$

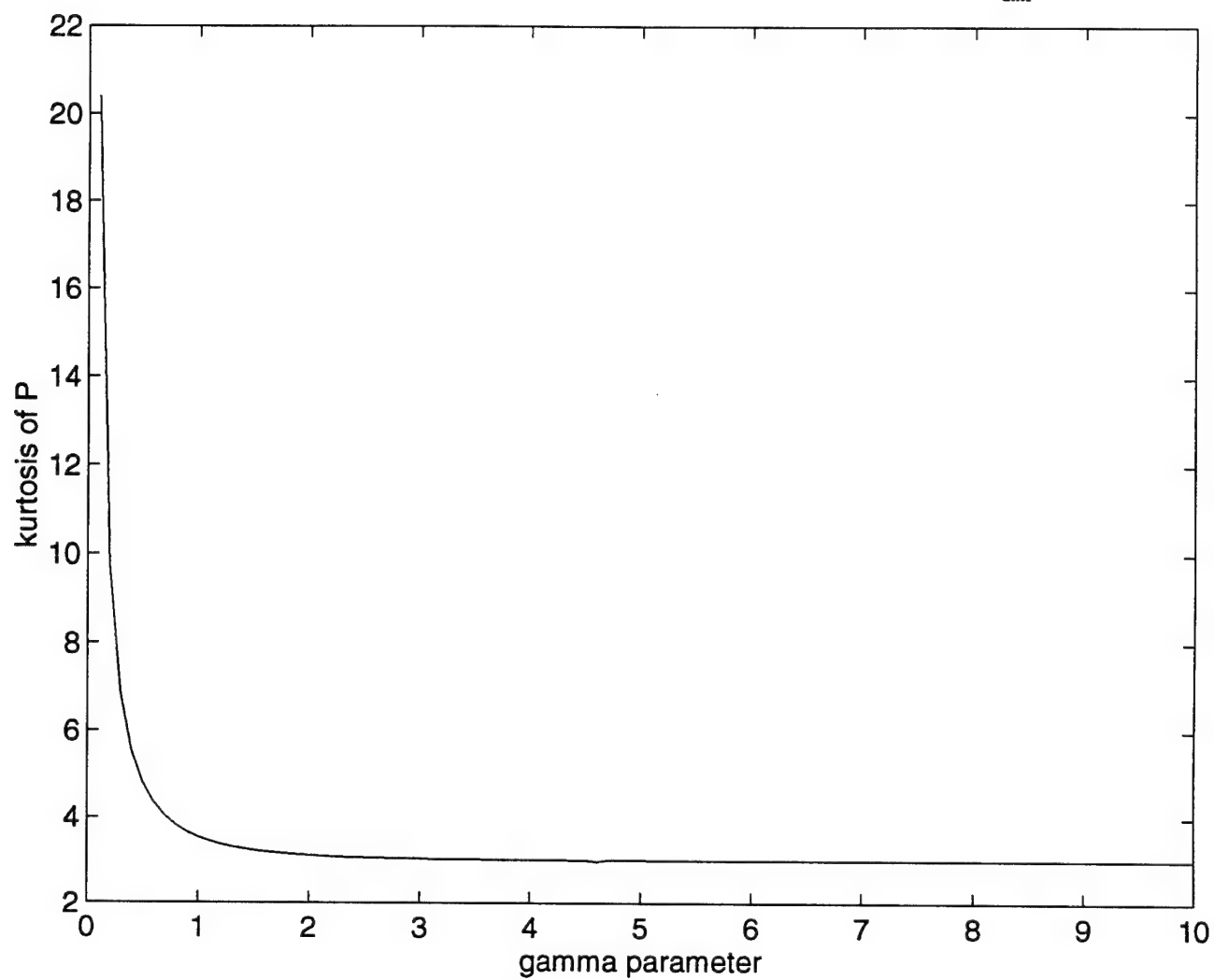
$$= \frac{3}{h^2\gamma^2(1-\rho^2)e^{h^2\gamma^2(1-\rho^2)}} \left\{ \frac{e^{-h^2\gamma^2(1-\rho^2)} - h^2\gamma^2(1-\rho^2)E_1[h^2\gamma^2(1-\rho^2)]}{E_1^2[h^2\gamma^2(1-\rho^2)]} \right\}$$

This implies that

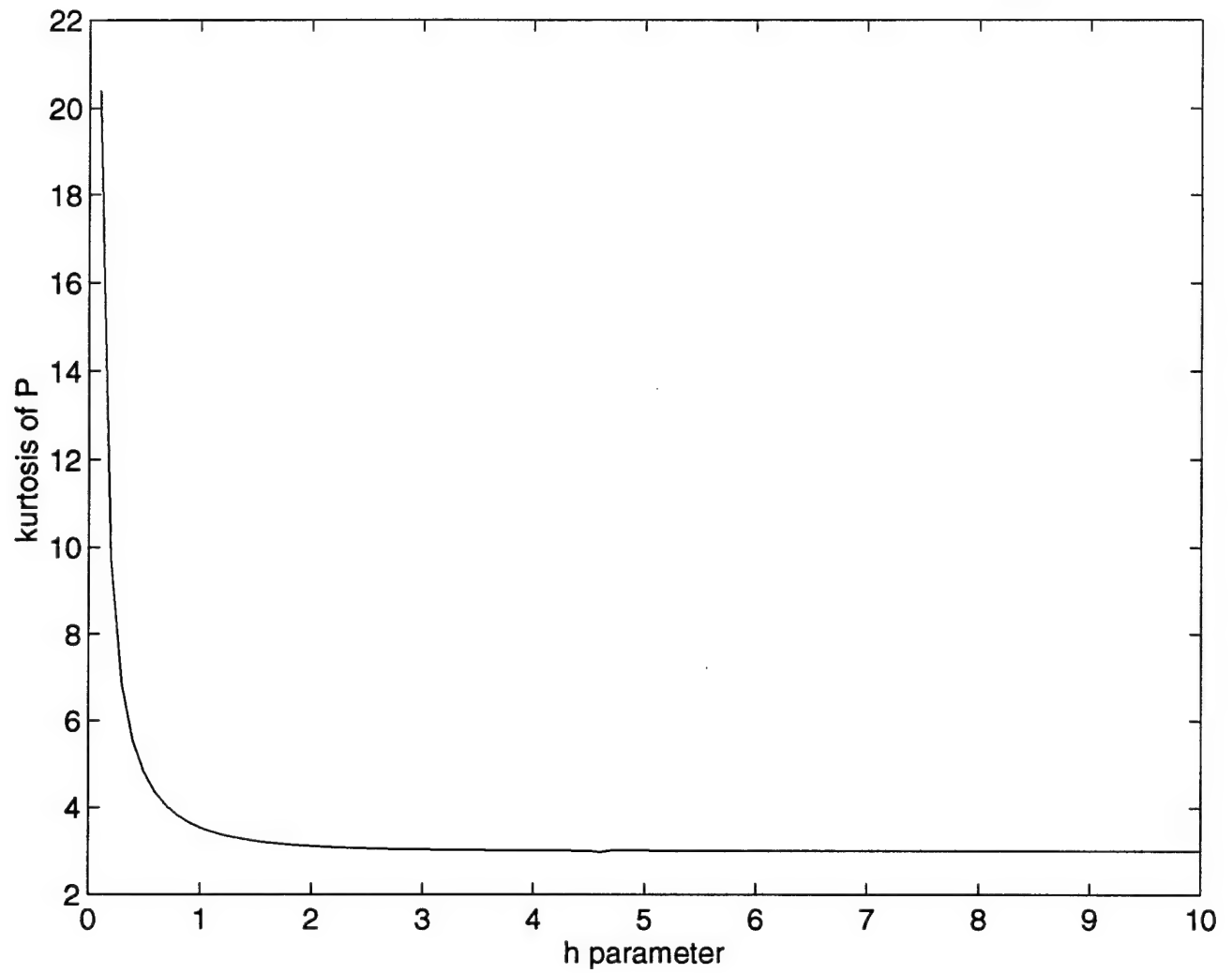
Kurtosis(P) =

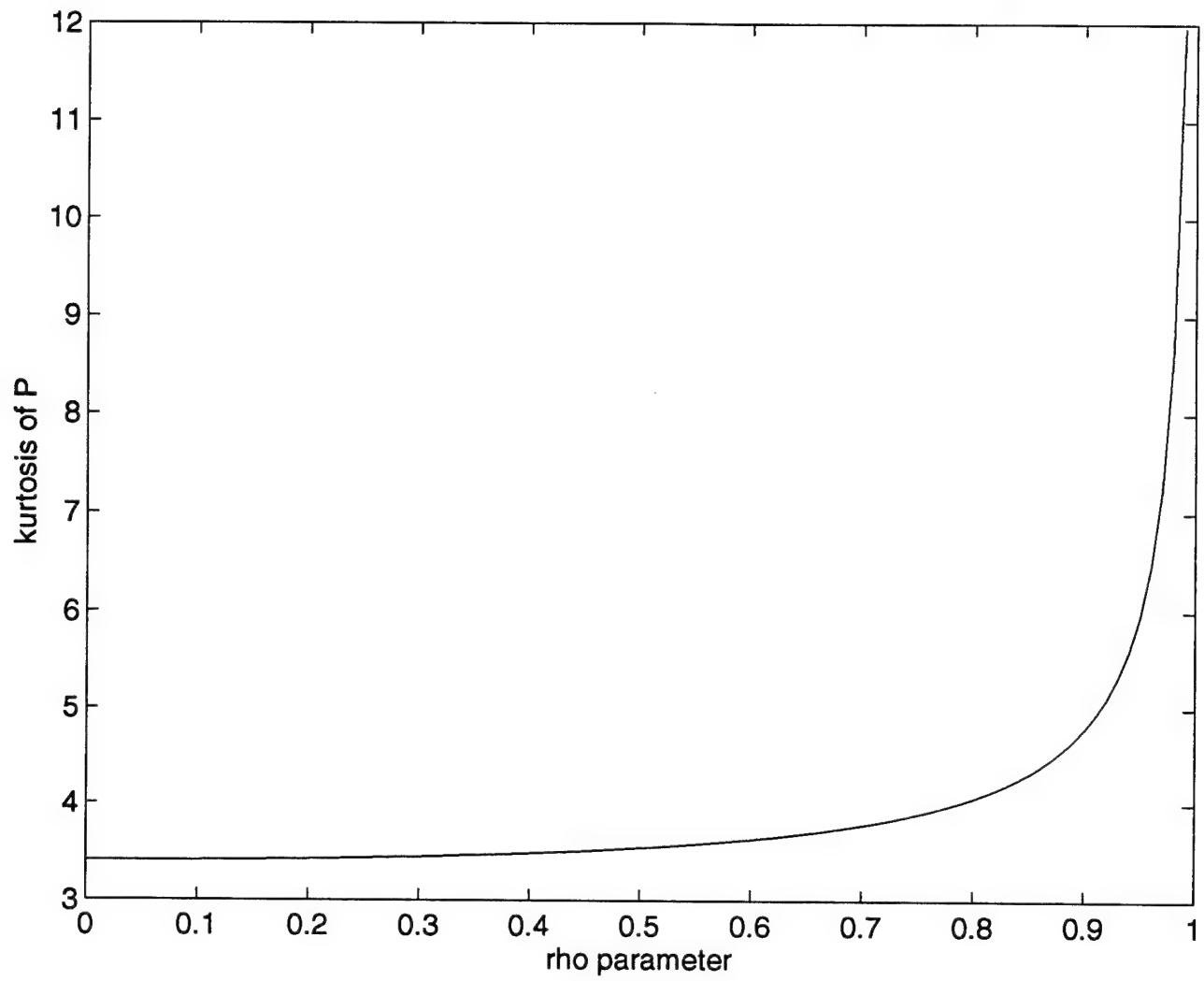
$$\frac{3}{h^2 \gamma^2 (1 - \rho^2) e^{h^2 \gamma^2 (1 - \rho^2)}} \left\{ \frac{e^{-h^2 \gamma^2 (1 - \rho^2)} - h^2 \gamma^2 (1 - \rho^2) E_1 [h^2 \gamma^2 (1 - \rho^2)]}{E_1^2 [h^2 \gamma^2 (1 - \rho^2)]} \right\} \quad (3.33)$$

This formula can be used to compare with the number, 3, the kurtosis of any normal distribution. What follows are some plots of the kurtosis of P versus different values for h, r, and g, respectively. Notice how they compare with the kurtosis of normality, namely 3. The kurtosis seems to come close to 3 for large values of h and g, while in the case of the r, correlation between X and U, the kurtosis is close but not equal to 3 for small values of r. The kurtosis grows large for small values of g and h, and it also grows large as r approaches 1. This shows that for large values of h and g and for small values of r, the kurtosis of the real part of the monopulse ratio has a kurtosis that is close to that of a normal distribution.

FIGURE 3-1. KURTOSIS OF P VS. THE  $\gamma$  PARAMETER



FIGURE 3-2. KURTOSIS OF P VS. THE  $h$  PARAMETER

FIGURE 3-3. KURTOSIS OF P VS. THE  $\rho$  PARAMETER

## CHAPTER 4

COMPARISON OF THE REAL PART OF THE MONOPULSE  
RATIO TO NORMALITY USING A STOCHASTIC DISTANCE MEASURE

Another method of giving a measurement of the closeness between the real part of the monopulse ratio,  $P$ , and a Gaussian distribution uses a "stochastic distance measure" between the density of  $P$  and the Gaussian density.

Given two probability density functions, say,  $f$  and  $g$ , the particular stochastic distance measure that will be used here is defined as a quantity,  $y$ , such that

$$y = \int_{-\infty}^{\infty} [f(p) - g(p)]^2 dp \quad (4.1)$$

For the case being considered, let  $f(p) = P_p(p)$ , the probability density function of the real part of the monopulse ratio, whose formula has already been derived earlier in this paper. Furthermore, let  $g(p) = n(p; \mu, \sigma^2)$  the normal distribution. Recall from any statistics or probability theory textbook that

$$n(p; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{(p - \mu)^2}{\sigma^2}\right) \quad (4.2)$$

Also, recall that for the real part of the monopulse ratio, the mean,  $\mu$ , is equal to  $\rho\gamma$ .

Using all the formulas mentioned above and leaving the variance,  $\sigma^2$ , as an unknown quantity, one can evaluate the stochastic distance measure integral. Since the stochastic distance measure must be numerically approximated on the computer, the lower and upper limits of integration needs to be replaced by two large numbers, say  $L$  and  $U$ . Hence, the numerical approximation of the stochastic distance measure has the following formula:

$$y(\sigma) = \int_{-L}^U [N(p; \rho\gamma, \sigma^2) - P_p(p)]^2 dp; \quad L, U \gg 0 \quad (4.3)$$

With this construction in mind, a numerical minimization algorithm can be used to approximate a  $\sigma$  such that  $y(\sigma)$  is minimized. This was done by programming  $y(\sigma)$  in the

MATLAB language, and a MATLAB routine known as FMIN was used to estimate  $\sigma$  such that  $y(\sigma)$  was minimized. The result was as follows:

**Given inputs:**  $\gamma = 1.$ ,  $\rho = 0.5$ ,  $h = 6.$ ,  $L = 10$ , and  $U = 12$

**Result:**  $\sigma = 0.1157$  and  $y(\sigma) = 3.6096 \times 10^{-7}$  (4.4)

Thus, for the  $\sigma=0.1157$ , which was the approximate value that minimized the stochastic distance measure, the value of the distance measure is as shown. The smallness of this number leads one to intuitively believe that for  $\sigma=0.1157$  and the given parameters,  $P_p(p)$  can be approximated accurately with a Gaussian density function.

## CHAPTER 5

### SOME COMMENTS CONCERNING FLUCTUATING TARGETS

In the next three sections to follow, a comparison is made between some of the results found in Reference 1 and those of References 2 and 3. In Reference 2, two cases of the target were considered, namely a nonfluctuating target scenario and a target whose signal, other than disturbance noise, fluctuates in accordance to a Rayleigh probability distribution. The comparative analysis to follow this section mainly deals with a Rayleigh fluctuating target. A review of the characteristics of the Rayleigh distribution can be found in Appendix C.

According to Reference 2 (page 626), for a Rayleigh fluctuating target,  $E[S]=E[D]=0$ . This was one of the assumptions for the thresholding case in Reference 1. Also, as will be shown in the next three sections to follow, the results for the variance and mean in the thresholding case in Reference 1 are equivalent to those found in References 2 and 3 for the real part of the monopulse ratio. Therefore, this implies that for the thresholding case with the covariance matrix as previously given, the real part of the target signal other than the noise disturbance of the monopulse radar in Reference 1 must implicitly also possess a Rayleigh distribution.

Note that in References 2 and 3, the monopulse ratio was defined only as the real part of the complex valued monopulse ratio as defined in Reference 1. Also, this real part is represented by  $R$  in References 2 and 3, while it is represented by  $P$  in both Reference 1 and this present report. Hence, the comparative study in the next three sections to follow shall only analyze properties of  $P$ , not  $Q$ .

## CHAPTER 6

## COMPARISON OF THE SIGNAL MEAN AND VARIANCE EQUATIONS

In this section, the equations for the mean and variance of the real part of monopulse ratio with thresholding as given in Reference 1 will be compared to those given in References 2 and 3. These formulas will be shown to be the same as long as any assumptions given in the references are properly taken into consideration.

As already stated, let  $D=X+iY$  be the difference channel,  $S=U+iV$  be the sum channel and  $P+iQ=D/S$  be the monopulse ratio. For simplicity in notation, let the expectation symbol for any random variable, say  $A$ , be expressed as

$$E[A] \equiv \bar{A} \quad (6.1)$$

Finally, let the symbol,  $\Delta$ , represent the event that the signal's amplitude exceeds the threshold, represented by  $R_0$ .

In Reference 1 (near the top of page 7-3), the mean of the real part of the monopulse ratio conditioned upon the event,  $\Delta$ , is given by the following equations:

$$E[P|\Delta] = \frac{\sigma_{\text{real}(D)}}{\sigma_{\text{real}(S)}} \rho = \frac{\sqrt{E[(X - \bar{X})^2]}}{\sqrt{E[(U - \bar{U})^2]}} \rho \quad (6.2)$$

According to Reference 2, Equation (40),

$$\begin{aligned} E[P|\Delta] &= E[R] = \rho(a/b) \\ \text{where } 2a^2 &\equiv E[|D - \bar{D}|^2] \text{ and } 2b^2 \equiv E[|S - \bar{S}|^2] \\ \Rightarrow E[P|\Delta] &= E[R] = \rho \frac{\sigma_D}{\sigma_S} \end{aligned} \quad (6.3)$$

This implies that

$$\begin{aligned}
 E[P|\Delta] &= \rho \frac{\sqrt{E[|X + iY - (\bar{X} + i\bar{Y})|^2]}}{\sqrt{E[|U + iV - (\bar{U} + i\bar{V})|^2]}} \\
 &= \rho \frac{\{E[(X - \bar{X})^2] + E[(Y - \bar{Y})^2]\}^{1/2}}{\{E[(U - \bar{U})^2] + E[(V - \bar{V})^2]\}^{1/2}}.
 \end{aligned}
 \tag{6.4}$$

Keep in mind that in deriving this equation, an assumption was made that the amplitude of the signal from the target fluctuated randomly with a Rayleigh probability distribution function. Also, according to the analysis in the previous section, the thresholding part of Reference 1, which is the part of that report being analyzed here, must also have a Rayleigh fluctuating target.

Note that in Reference 1, an assumption was made that the variances of the real and imaginary parts of the sum channel are equal. Similarly, the variances of the real and imaginary parts of the difference channels are also assumed to be equal. In this case the above results become

$$\begin{aligned}
 E[P|\Delta] &= \rho \frac{\{2E[(X - \bar{X})^2]\}^{1/2}}{\{2E[(U - \bar{U})^2]\}^{1/2}} \\
 &= \rho \frac{\sigma_{\text{real}(D)}}{\sigma_{\text{real}(S)}}.
 \end{aligned}
 \tag{6.5}$$

This is exactly the same result as given in Reference 1. Hence, given the assumptions expressed in the two papers being compared, the equations for  $E[R|R_0]$  as given by References 1 and 2 are identical.

Now again assuming a fluctuating amplitude target signal with a threshold of  $R_0$ , the variances as reported in the two papers will be shown to be the same. The formula in Seifer's analysis will be given first and then manipulated and compared with the expression in Reference 1. Assumptions that are used in the derivation to follow will be applied.

In Equation (33b) of Reference 2, the expression for the signal variance conditioned on the threshold is given by

$$\sigma^2 [P|\Delta] = \frac{1}{2} \mu^2 \exp\left(-\frac{l_0^2}{\chi + 1}\right) E_1\left(-\frac{l_0^2}{\chi + 1}\right) \text{ where } \chi = \frac{(E[|F|])^2}{\sqrt{E[|\eta|^2]}},$$

$$l_0 = \text{normalized threshold}, \frac{R_0}{\sqrt{E[|\eta|^2]}}, \eta = \text{sum channel signal noise},$$

$$E_1(x) = \int_x^\infty \frac{1}{t} e^{-t} dt, \quad (6.6)$$

$$\mu = \frac{a}{b} \sqrt{1 - \rho^2 - \zeta^2}, \rho + i\zeta = \text{correlation}(D, S)$$

$\chi$  as defined above is interpreted as a "signal-to-noise ratio."

Since a Rayleigh distributed fluctuating target is assumed, the sum channel signal can be considered to be the sum of the noise plus the randomly fluctuating actual sum channel signal Sinformation itself,  $F$ . Hence,  $S = F + \eta$ , which is shown in Equation (28) of Reference 2. Note that must also be  $F + \eta$  in Reference 1 for the thresholding case with the given covariance matrix due to the argument in the previous section. According to Equation (31) of Reference 2, the following relationship can be shown:

$$l_0^2 = \frac{R_0^2 (\chi + 1)}{2b^2}, \text{ recalling } 2b^2 = E[|S - \bar{S}|^2] \quad (6.7)$$

$2b^2$  can be shown to be equivalent to the sum of the variances of the real and imaginary parts of the sum channel in the following way:

$$\begin{aligned} E[|S - \bar{S}|^2] &= E[|(u + iv) - (\bar{u} + i\bar{v})|^2] \\ &= E[|(u - \bar{u}) + i(v - \bar{v})|^2] \\ &= E[(u - \bar{u})^2 + (v - \bar{v})^2] \Rightarrow E[|S - \bar{S}|^2] = \sigma_u^2 + \sigma_v^2 \end{aligned} \quad (6.8)$$

$$\Rightarrow \frac{l_0^2}{\chi + 1} = \frac{R_0^2}{\sigma_u^2 + \sigma_v^2} \Rightarrow \quad (6.9)$$

$$\sigma^2 [P|\Delta] = \frac{a^2}{2b^2} (1 - \rho^2 - \zeta^2) \exp\left(-\frac{R_0^2}{\sigma_u^2 + \sigma_v^2}\right) E_1\left(-\frac{R_0^2}{\sigma_u^2 + \sigma_v^2}\right) \quad (6.10)$$



$$\therefore \text{setting } \frac{a^2}{b^2} = \frac{\sigma_D^2}{\sigma_S^2}, \quad (6.11)$$

$$\sigma^2 [P|\Delta] = \frac{\sigma_D^2}{2\sigma_S^2} (1 - \rho^2 - \zeta^2) \exp\left(\frac{R_0^2}{\sigma_u^2 + \sigma_v^2}\right) E_1\left(\frac{R_0^2}{\sigma_u^2 + \sigma_v^2}\right)$$

The variance formula, which has a slight error, is given in Equation (7.17) of Reference 1. The corrected version of this variance formula is derived in Appendix A and is as follows:

$$\therefore \text{Var}(P) \equiv \sigma^2 [R|\Delta] = \frac{1}{2} \gamma^2 (1 - \rho^2) e^{h^2 \gamma^2 (1 - \rho^2)} E_1[h^2 \gamma^2 (1 - \rho^2)], \quad (6.12)$$

$$\text{where } h = \frac{R_0}{\sigma_x \sqrt{2(1 - \rho^2)}}.$$

The h can be manipulated as follows:

$$h^2 \gamma^2 (1 - \rho^2) = \frac{R_0^2}{2\sigma_x^2 (1 - \rho^2)} [\gamma^2 (1 - \rho^2)] = \frac{R_0^2 \sigma_x^2}{2\sigma_x^2 \sigma_u^2} = \frac{R_0^2}{2\sigma_u^2},$$

$$\text{recalling that } \gamma \equiv \frac{\sigma_x}{\sigma_u}; \quad (6.13)$$

$$\therefore \sigma^2 [P|\Delta] = \frac{\sigma_x^2}{2\sigma_u^2} (1 - \rho^2) \exp\left(\frac{R_0^2}{2\sigma_u^2}\right) E_1\left(\frac{R_0^2}{2\sigma_u^2}\right)$$

Since X and U are the real parts of the difference and sum channels respectively,

$$\sigma_x = \sigma_{\text{real}(D)} \text{ and } \sigma_u = \sigma_{\text{real}(S)} \quad (6.14)$$

Now a comparison between the conditional variance formula from Reference 2 and the modified formula for the conditional variance from Reference 1 can be made. In order to make this comparison, one needs to recall from a previous derivation that

$$\sigma_S^2 = E[|S - \bar{S}|^2] = \sigma_u^2 + \sigma_v^2 \quad (6.15)$$

By an identical argument,

$$\sigma_D^2 = E[|D - \bar{D}|^2] = \sigma_x^2 + \sigma_y^2 \quad (6.16)$$

In Reference 1, another assumption was that the real and imaginary parts of the variance of D are the same. Similarly, the real and imaginary parts of the variance of S are assumed to be the same in this report. This implies that

$$\sigma_D^2 = 2\sigma_x^2 \text{ and } \sigma_S^2 = 2\sigma_u^2 \quad (6.17)$$

Furthermore, Equation (5.1) of Reference 1 indicates that correlations between parts of S and D do not contain any imaginary parts. Instead, the correlation between the real part of S and the real part of D as well as the correlation between the imaginary part of S and D are represented by the same number,  $\sigma$ , and cross correlations between real and imaginary parts are assumed to be zero. These assumptions imply that  $\zeta$ , which is the imaginary part of correlation (S, D), must be 0. Substituting all these results, the two equations for  $\sigma^2[R|\Delta]$  show that the conditional variances given in Reference 2 is equivalent to the corrected form of the conditional variance in Reference 1 provided that the given assumptions are properly applied.

This derivation is actually more general than necessary since  $E[S]$  and  $E[D]$  are known to be zero for a Rayleigh fluctuating target. However, this result was not substituted, which would only slightly simplify the derivation, in order to avoid confusion on the meaning of the variances.

As a byproduct of this derivation, notice that given the above stated assumptions,

$$h^2 \gamma^2 (1 - \rho^2) = \frac{l_0^2}{\chi + 1} \text{ since } h^2 \gamma^2 (1 - \rho^2) = \frac{R_0^2}{2\sigma_u^2} = \frac{l_0^2}{\chi + 1} \quad (6.18)$$

given that  $\sigma_u = \sigma_v$ . This result will be used in the analysis of the conditional variance with the presence of noise jamming.

## CHAPTER 7

COMPARISON OF THE SIGNAL MEAN AND VARIANCE  
EQUATIONS IN THE PRESENCE OF NOISE JAMMING

Another scenario considered in Reference 2 was the case where noise jamming also disturbed the radar target signal. This noise represents an intentional attempt to deceive the radar from accurately tracking the target.

Several assumptions were made for the analysis in reference to the jamming by Seifer in Reference 2 (page 633). These assumptions are as follows:

- jamming comes from a single point source.
- the single point source is within the principal lobe of the receiving antenna's sum pattern and also copolarized with the antenna response there.
- jamming dominates thermal noise of the receiver.

The analysis of the means of the two papers will now be given. As previously given, the formula from the conditional mean for a Rayleigh fluctuating target according to Equation (40) of Reference 2 is

$$E[P|\Delta] = E[P] = \rho \frac{a}{b} \quad (7.1)$$

while the same expectation according to Reference 1 (near the top of page 7-3) is

$$E[P|\Delta] = \frac{\sigma_{\text{real}(D)}}{\sigma_{\text{real}(S)}} \rho \quad (7.2)$$

such that, as before,  $\rho$  is the real part of correlation of D and S for both papers.

Let  $D_J$  be the complex-valued quantity representing the contribution on the difference channel, and let  $G_J$  be the same on the sum channel. Let  $r_J$  be  $D_J/G_J$ . According to Equation (72) of Reference 2,

$$E[P|\Delta] = \rho \frac{a}{b} = \frac{\chi \hat{r} + r_J}{\chi + 1} \Rightarrow \quad (7.3)$$

$$E[P|\Delta] - \hat{r} = \frac{r_J - \hat{r}}{\chi + 1}$$

where  $\hat{r}$  is defined as the real part of the monopulse ratio without random disturbance. Since the meaning of  $a$  and  $b$  is the same as previously defined, and the equations for  $E[R|\Delta]$  in the two reports are equivalent under the assumptions given, this equation for the jamming case must also apply to the model as given in Reference 1.

The variance equations for the two reports under noise jamming will now be analyzed. Recall that, with the given assumptions,

$$\begin{aligned} \mu^2 &= \frac{a^2}{b^2} \left( \sqrt{1 - \rho^2 - \zeta^2} \right)^2 = \frac{2a^2}{2b^2} (1 - \rho^2 - \zeta^2) \\ &= \frac{\sigma_x^2 + \sigma_y^2}{\sigma_u^2 + \sigma_v^2} (1 - \rho^2) = \frac{2\sigma_x^2}{2\sigma_u^2} (1 - \rho^2) = \gamma^2 (1 - \rho^2) \Rightarrow \end{aligned} \quad (7.4)$$

$$\frac{1}{2} \mu^2 = \frac{1}{2} \gamma^2 (1 - \rho^2)$$

The result as shown in Equation (72) of Reference 2 implies that

$$\frac{1}{2} \mu^2 = \frac{1}{2} \frac{\chi}{(\chi + 1)^2} |r_J - \hat{r}|^2 \quad (7.5)$$

$$\therefore \frac{1}{2} \gamma^2 (1 - \rho^2) = \frac{1}{2} \frac{\chi}{(\chi + 1)^2} |r_J - \hat{r}|^2$$

Substituting this result into the equation for the conditional variance according to Reference 1 gives

$$\begin{aligned} \sigma^2 [P|\Delta] &= \\ \frac{1}{2} \frac{\chi}{(\chi + 1)^2} |r_J - \hat{r}|^2 \exp(h^2 \gamma^2 (1 - \rho^2)) E_1(h^2 \gamma^2 (1 - \rho^2)) \end{aligned} \quad (7.6)$$

Substituting the formula for  $h^2\gamma^2(1-\rho^2)$  in Equation (6.18) into (7.6) implies that

$$\sigma^2 [P|\Delta] = \frac{1}{2} \frac{\chi}{(\chi+1)^2} |r_J - \hat{r}|^2 \exp\left(\frac{l_0^2}{\chi+1}\right) E_1\left(\frac{l_0^2}{\chi+1}\right) \Rightarrow \quad (7.7)$$

$$\frac{\sigma[P|\Delta]}{|r_J - \hat{r}|} = \sqrt{\frac{1}{2} \frac{\chi}{(\chi+1)^2} \exp\left(\frac{l_0^2}{\chi+1}\right) E_1\left(\frac{l_0^2}{\chi+1}\right)}$$

which is the same as the formula in Equation (76) of Reference 2.

Thus, the conclusion of this derivation is the fact that the conditional variances equations of Reference 1 and Reference 2 are equivalent under the given hypotheses even in the presence of a single point noise jamming.

## CHAPTER 8

COMPARISON OF THE SIGNAL MEAN AND  
VARIANCE EQUATIONS FOR TRACKING IN NOISE

In July, 1994, Arnold Seifer's correspondence, supplementing his work shown in Reference 2 appeared in Reference 3. Among other new assumptions, two of them are the hypotheses that no noise jamming is present and random disturbance noise between the sum channel and difference channel are uncorrelated.

This lack of correlation between the random disturbance of the sum and difference channel signal does not necessarily mean that S and D are uncorrelated, but only that the random disturbance noise portions of S and D are uncorrelated. Hence,  $\rho$  is not necessarily 0 although, recalling one of the hypothesis of Reference 1, the imaginary part of the correlation between D and S (namely,  $\zeta$ ) is assumed to be 0.

The derivations in Reference 3 for the conditional mean and the conditional variance are based upon the assumption of the lack of correlation between the random disturbance of the sum and difference channels. This assumption was used to obtain the following relation, which can be found in Equation (16) of Reference 3

$$\frac{a}{b}(\rho + i\zeta) = \frac{\chi}{\chi + 1} \hat{r}; \hat{r} \text{ real by assumption} \quad (8.1)$$

Also, as Seifer stated on the same page, the monopulse ratio without the disturbance noise is by definition real. Therefore, this fact along with the assumption in Reference 1 that  $\zeta=0$ , each implies that

$$\frac{a}{b}\rho = \frac{\chi}{\chi + 1} \hat{r} \quad (8.2)$$

which was also shown in Reference 3 on the same page. These two relations were the only ones that were used in Reference 3 for deriving new formulas for the conditional mean and variances for the Rayleigh fluctuating case conditioned upon the event,  $R_0$ .

Since the assumption of 0 correlation between the random disturbances of D and S, along with the above equations derived from this assumption, does not violate any relations given in Reference 1, these results in Reference 3 for the conditional mean and conditional variance of a Rayleigh fluctuating target are still equivalent to the formulas derived in Reference 1. As noted above, jamming was not considered in Reference 3.

## CHAPTER 9

**EQUIVALENCE OF A COMPLEX VALUED RANDOM  
VARIABLE TO A TWO-DIMENSIONAL RANDOM VECTOR**

This paper has dealt mainly with the real part of the monopulse ratio as a random variable. Combining the random variable,  $P$ , with the imaginary part,  $Q$ , the monopulse ratio becomes a complex valued random variable  $P+iQ$ . Another obvious way of representing this is by using the ordered pair  $(P,Q)$  or its corresponding random vector (the transpose of  $(P,Q)$ ). Although, intuitively, complex and vector representations seem clearly statistically equivalent, complex numbers are mathematically treated differently from ordered pairs. This difference may cause some people to still question whether the two representations are equivalent. Since the difference in the mathematical treatment between complex numbers and ordered pair representations are not considered in determining whether the two entities are probabilistically the same, they are in fact equivalent. The explanation to follow gives a justification for this equivalence. This explanation will be in the form of a theorem.

Theorem 1: Given a complex valued random variable,  $X+iY$ , there exists a two-dimensional random vector, namely  $(X, Y)^T$ , (or bivariate distributed ordered pair, namely  $(X,Y)$ ) that is statistically equivalent to the complex valued random variable.

A proof of this theorem is given in Appendix D. With this theorem in mind, we can use the three forms of the bivariate random variable in  $X$  and  $Y$  as given in the statement of the theorem, interchangeably.

For a precise measure-theoretic definition of a random variable, one is referred to any textbook on fundamental probability theory that is based upon a measure-theoretic approach.

Due to this equivalence that is now justified, the random variable  $P+iQ$  and the ordered pair  $(P, Q)$  as well as  $(P, Q)^T$  can be discussed interchangeably as a model of the real and imaginary portions of the radar monopulse ratio. This should help clarify any ambiguity.

## CHAPTER 10

### SUMMARY

Several results have been produced by the investigative project described here. A closed form formula of the probability density function of,  $P$ , the real part of the monopulse ratio of a monopulse radar was derived. Furthermore, two methods of comparing this probability density function with a Gaussian density were derived. One was based upon kurtosis, and the other one was based upon a stochastic distance measure. A third method that could be used to check closeness to normality is to compare the areas under the curve of the tail regions of the probability density function of  $P$  and a normal probability density function. This can be investigated in the future. We showed that the kurtosis came close to that of a normal distribution for large values of the normalized threshold,  $h$ , large values of  $\gamma = \sigma_X / \sigma_U$ , and small values of the correlation between  $X$  and  $U$  (namely  $\rho$ ). We also showed that under certain assumptions, the stochastic distance measure of the normal distribution and the distribution of the real part of the monopulse ratio is small. A comparative analysis was performed in relation to some of the formulas for the mean and variance of  $P$  as presented in Reference 1 with those of References 2 and 3. We showed that given the correct hypothesis, the formulas were equivalent in spite of the fact that the equations had different forms and were derived from different approaches.

In conclusion, a closed form formula for the distribution of  $P$ , is now derived and available for use in evaluating the accuracy of monopulse radars. We also have derived two tools in comparing the real part of the monopulse ratio distribution with the normal distribution. Also, using these tools, we showed that given certain assumptions,  $P$  can be approximated as a normally distributed random variable. Finally, we showed that despite the different approaches of Reference 1 as compared to References 2 and 3, the formulas for the variance and mean of  $P$  with and without jamming are equivalent.



## REFERENCES

1. Groves, G. W. and Blair, W. D., *Statistical Studies of the Monopulse Ratio*, NSWCDD/TR-94/97, Oct 1994, Naval Surface Warfare Center, Dahlgren Division, Dahlgren, VA.
2. Seifer, A. D., "Monopulse-Radar Angle Tracking in Noise or Noise Jamming," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 28, No. 3, Jul 1992, p. 622-637.
3. Seifer, A. D., Monopulse-Radar Angle Measurement in Noise, *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 30, No. 3, Jul 1994, p. 950-957.
4. Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series, and Products* (Corrected and Enlarged Edition), Academic Press, New York, 1980.

**APPENDIX A**

**DERIVATION OF THE VARIANCE OF THE REAL  
PART OF THE MONOPULSE RATIO WITH THRESHOLDING**

In the section on kurtosis, the variance of the real part of the monopulse ratio was given in Equation (3.29) as

$$E[t^2] = \frac{\gamma^2 (1 - \rho^2)}{2} \exp(h^2 \gamma^2 (1 - \rho^2)) E_1[h^2 \gamma^2 (1 - \rho^2)] \quad (A.1)$$

Recall that  $t$  was defined to be  $P - E[P]$ , and  $E[P] = \rho\gamma$ . Also recall that this formula differed slightly from the one shown in Reference 1 of the main text.\* What follows is a derivation of the formula for the variance as given here. Recall from the section on kurtosis that according to Equation (7.16) of Reference 1, the variance can be obtained from the following integral.

$$\begin{aligned} E[t^2] &= \frac{\gamma^2 (1 - \rho^2)}{\pi} * \\ &\int_0^{2\pi} \cos^2 \theta d\theta \int_0^\infty R^3 dR \left[ \frac{h^2 [R^2 + \gamma^2 (1 - \rho^2)] + 1}{[R^2 + \gamma^2 (1 - \rho^2)]^2} \right] \exp[-h^2 R^2] \\ &= \gamma^2 (1 - \rho^2) \int_0^\infty R^3 dR \left[ \frac{h^2 [R^2 + \gamma^2 (1 - \rho^2)] + 1}{[R^2 + \gamma^2 (1 - \rho^2)]^2} \right] \exp[-h^2 R^2] \end{aligned} \quad (A.2)$$

In performing this integration, we need to do a change of variable by letting  $u = R^2$ . Then  $du = 2R dR$ , and  $R dR = (1/2) du$ . Substituting,

$$E[t^2] = \frac{1}{2} \gamma^2 (1 - \rho^2) \int_0^\infty u du \left[ \frac{h^2 [u + \gamma^2 (1 - \rho^2)] + 1}{[u + \gamma^2 (1 - \rho^2)]^2} \right] \exp[-h^2 u] \quad (A.3)$$

Now changing variables again, let  $a = u + \gamma^2 (1 - \rho^2)$ , implying that  $du = da$  and  $u = a - \gamma^2 (1 - \rho^2)$ . This gives

$$\begin{aligned} E[t^2] &= \\ \frac{1}{2} \gamma^2 (1 - \rho^2) e^{h^2 \gamma^2 (1 - \rho^2)} &\int_{\gamma^2 (1 - \rho^2)}^\infty [a - \gamma^2 (1 - \rho^2)] da \left( \frac{h^2 a + 1}{a^2} \right) e^{-h^2 a} = \end{aligned} \quad (A.4)$$

$$\begin{aligned} K \left\{ \int_{\gamma^2 (1 - \rho^2)}^\infty \left( \frac{h^2 a + 1}{a} \right) e^{-h^2 a} da - \gamma^2 (1 - \rho^2) \int_{\gamma^2 (1 - \rho^2)}^\infty \left( \frac{h^2 a + 1}{a^2} \right) e^{-h^2 a} da \right\} \\ \text{for } K = (1/2) \gamma^2 (1 - \rho^2) \exp(h^2 \gamma^2 (1 - \rho^2)). \end{aligned} \quad (A.5)$$

\* Groves, G. W. and Blair, W. D., *Statistical Studies of the Monopulse Ratio*, NSWCDD/TR-94/97, Oct 1994, Naval Surface Warfare Center, Dahlgren Division, Dahlgren, VA.

Multiplying out the terms,

$$\begin{aligned}
 E[t^2] = K \{ & h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2\alpha} d\alpha + \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha}\right) e^{-h^2\alpha} d\alpha - \\
 & \gamma^2(1-\rho^2) h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha}\right) e^{-h^2\alpha} d\alpha \\
 & - \gamma^2(1-\rho^2) \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha^2}\right) e^{-h^2\alpha} d\alpha \} \Rightarrow
 \end{aligned} \tag{A.6}$$

$$\begin{aligned}
 E[t^2] = & K \{ h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2\alpha} d\alpha + [1 - \gamma^2(1-\rho^2) h^2] \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha}\right) e^{-h^2\alpha} d\alpha \\
 & - \gamma^2(1-\rho^2) \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha^2}\right) e^{-h^2\alpha} d\alpha \}
 \end{aligned} \tag{A.7}$$

Now consider each integral term separately.

$$\int_{\gamma^2(1-\rho^2)}^{\infty} e^{-h^2\alpha} d\alpha = \frac{e^{-h^2\alpha}}{-h^2} \Big|_{\gamma^2(1-\rho^2)}^{\infty} = \frac{e^{-h^2\gamma^2(1-\rho^2)}}{h^2} \tag{A.8}$$

Now consider the integral,

$$\int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha}\right) e^{-h^2\alpha} d\alpha \tag{A.9}$$

Integration by parts will be used for this integration.

$$\begin{aligned}
 \text{Let } w = e^{-h^2\alpha} \text{ and } dv = \alpha^{-2} d\alpha. \therefore dw = -h^2 e^{-h^2\alpha} d\alpha \text{ and } v = -\alpha^{-1} \Rightarrow \\
 \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha^2}\right) e^{-h^2\alpha} d\alpha = -\alpha^{-1} e^{-h^2\alpha} \Big|_{\gamma^2(1-\rho^2)}^{\infty} - h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha}\right) e^{-h^2\alpha} d\alpha \\
 = \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha}\right) e^{-h^2\alpha} d\alpha
 \end{aligned} \tag{A.10}$$

Now suppose  $s=h^2\alpha$ . Then  $ds=h^2d\alpha$  and  $d\alpha=ds/h^2$ . Substituting,

$$\frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{1}{\alpha}\right) e^{-h^2\alpha} d\alpha = \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - \quad (A.11)$$

$$\begin{aligned} h^2 \gamma^2(1-\rho^2) \int_{\gamma^2(1-\rho^2)}^{\infty} \left(\frac{e^{-s}}{s/h^2}\right) ds &= \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 \int_{h^2\gamma^2(1-\rho^2)}^{\infty} \left(\frac{e^{-s}}{s}\right) ds \\ &= \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 E_1[h^2\gamma^2(1-\rho^2)] \end{aligned} \quad (A.12)$$

$$\therefore \int_{\gamma^2(1-\rho^2)}^{\infty} \frac{1}{\alpha^2} e^{-h^2\alpha} d\alpha = \frac{e^{-h^2\gamma^2(1-\rho^2)}}{\gamma^2(1-\rho^2)} - h^2 E_1[h^2\gamma^2(1-\rho^2)]$$

Finally, consider the integral term,

$$\int_{\gamma^2(1-\rho^2)}^{\infty} \frac{1}{\alpha} e^{-h^2\alpha} d\alpha \quad (A.13)$$

Let  $t=h^2\alpha$ . Then  $dt=h^2d\alpha$  and  $d\alpha=dt/h^2$ . This implies that

$$\begin{aligned} \int_{\gamma^2(1-\rho^2)}^{\infty} \frac{1}{\alpha} e^{-h^2\alpha} d\alpha &= \int_{h^2\gamma^2(1-\rho^2)}^{\infty} \frac{1}{h^2\gamma^2(1-\rho^2) t / h^2} e^{-t} \frac{dt}{h^2} \\ &= \int_{h^2\gamma^2(1-\rho^2)}^{\infty} \frac{e^{-t}}{t} dt \equiv E_1[h^2\gamma^2(1-\rho^2)] \end{aligned} \quad (A.14)$$

Inserting the formulas for these three integrals into the equation for  $E[t^2]$ ,

$$\begin{aligned}
 E[t^2] &= K \left\{ h^2 \left( \frac{e^{-h^2 \gamma^2 (1-\rho^2)}}{h^2} \right) + [1 - \gamma^2 (1-\rho^2) h^2] * \right. \\
 &\quad E_1 [h^2 \gamma^2 (1-\rho^2)] - \gamma^2 (1-\rho^2) \left[ \frac{e^{-h^2 \gamma^2 (1-\rho^2)}}{\gamma^2 (1-\rho^2)} - \right. \\
 &\quad \left. h^2 E_1 (h^2 \gamma^2 (1-\rho^2)) \right] \} = K \{ E_1 [h^2 \gamma^2 (1-\rho^2)] \} \\
 &= \frac{1}{2} \gamma^2 (1-\rho^2) e^{h^2 \gamma^2 (1-\rho^2)} E_1 [h^2 \gamma^2 (1-\rho^2)]
 \end{aligned} \tag{A.15}$$

$$\therefore \text{Var}(P) = E[t^2] = \frac{1}{2} \gamma^2 (1-\rho^2) e^{h^2 \gamma^2 (1-\rho^2)} E_1 [h^2 \gamma^2 (1-\rho^2)] \tag{A.16}$$

This is the closed form formula for the variance of P. Keep in mind that this is a conditional variance, conditioned upon the event that the sum channel signal amplitude exceeds the threshold.

**APPENDIX B**

**A GENERAL JOINT DISTRIBUTION OF THE MONOPULSE RATIO**

This document mainly dealt with the radar monopulse ratio with a specific type of covariance matrix between S and D and with the condition of thresholding. A generalized bivariate distribution of P and Q was given in Reference 1 of the main text\* (page 6-3). The equations are

$$P_{P,Q}(p, q) = \frac{K e^{-M}}{2\pi\Delta^2 \sqrt{-|\Sigma|\Delta}}, \text{ where}$$

$$\Delta \equiv B^2 - 4AC; M \equiv F + \Delta^{-1} (AE^2 - BDE + CD^2); K \equiv B^2 D^2 + B^2 E^2$$

$$-4BCDE + 4C^2 D^2 - 4ABDE + 4A^2 E^2 + 8A^2 C - 2B^2 C + 8AC^2 - 2AB^2$$

$$A \equiv \frac{1}{2} a_{11} p^2 + a_{12} pq + a_{13} p + \frac{1}{2} a_{22} q^2 + a_{23} q + \frac{1}{2} a_{33}$$

$$B \equiv -a_{11} pq + a_{12} (p^2 - q^2) - a_{13} q + a_{14} p + a_{22} pq + a_{23} p + a_{24} q + a_{34}$$

$$C \equiv \frac{1}{2} a_{11} q^2 - a_{12} pq - a_{14} q + \frac{1}{2} a_{22} p^2 + a_{24} p + \frac{1}{2} a_{44}$$

$$D \equiv -a_{11} \bar{x}p - a_{12} (\bar{y}p + \bar{x}q) - a_{13} (\bar{u}p + \bar{x}) - a_{14} \bar{v}p - a_{22} \bar{y}q - a_{23} (\bar{u}q + \bar{y})$$

$$-a_{24} \bar{v}q - a_{33} \bar{u} - a_{34} \bar{v}$$

$$E \equiv a_{11} \bar{x}q - a_{12} (\bar{x}p - \bar{y}q) + a_{13} \bar{u}q - a_{14} (\bar{x} - \bar{v}q) - a_{22} \bar{y}p - a_{23} \bar{u}p$$

$$-a_{24} (\bar{v}p + \bar{y}) - a_{34} \bar{u} - a_{44} \bar{v}$$

$$F \equiv \frac{1}{2} a_{11} \bar{x}^2 + a_{12} \bar{x}\bar{y} + a_{13} \bar{x}\bar{u} + a_{14} \bar{x}\bar{v} + \frac{1}{2} a_{22} \bar{y}^2 + a_{23} \bar{y}\bar{u} + a_{24} \bar{y}\bar{v}$$

$$+ \frac{1}{2} a_{33} \bar{u}^2 + a_{34} \bar{u}\bar{v} + \frac{1}{2} a_{44} \bar{v}^2$$

$$\Sigma^{-1} \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$$

(B.1)

\* Groves, G. W. and Blair, W. D., *Statistical Studies of the Monopulse Ratio*, NSWCDD/TR-94/97, Oct 1994, Naval Surface Warfare Center, Dahlgren Division, Dahlgren, VA.



This generalized bivariate probability density function was empirically tested by numerically doubly integrating the function with respect to p and q over the Euclidean plane to see whether the integration resulted in a value of 1. The input parameters were chosen to be as follows:

$$\Sigma^{-1} = \begin{pmatrix} 100 & 2 & 1.5 & 3 \\ 2 & 100 & -3 & 3.5 \\ 1.5 & -3 & 200 & 4.5 \\ 3 & 3.5 & 4.5 & 200 \end{pmatrix}; \bar{x} = 10, \bar{y} = 5, \bar{u} = 12, \bar{v} = -6 \quad (\text{B.2})$$

The programming for the numerical integration was performed in the MATLAB computer package language. This numerical double integration indeed produced a result of 1. In other words,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{K e^{-M}}{2 \pi \Delta^2 \sqrt{-|\Sigma| \Delta}} dp dq \approx 1 \quad (\text{B.3})$$

Since the input parameters were arbitrarily chosen, this further supports the correctness of the generalized probability density function as derived in Reference 1 of the main text.

**APPENDIX C**  
**THE RAYLEIGH DISTRIBUTION**

A Rayleigh probability function with a parameter of  $\sigma$  on a random variable  $z$  is defined on the non-negative real line as follows:

$$P_Z(z; \sigma) = \begin{cases} \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}} \quad \forall z \in [0, \infty) \\ 0 \quad \forall z \text{ elsewhere} \end{cases} \quad (C.1)$$

This distribution is very useful in modeling physical phenomena since it represents the magnitude of the vector sum of two independent and zero mean normally distributed random variables with the same variance. This fact can be more clearly stated when written in a form of a mathematical theorem.

Theorem 1A: Let  $X$  and  $Y$  be independent, zero mean, normally distributed random variables with the same standard deviation,  $\sigma$ . Then the transformation

$$Z = \sqrt{X^2 + Y^2} \quad (C.2)$$

has a Rayleigh distribution with parameter  $\sigma$ .

Proof: Since  $X$  and  $Y$  are independent normally distributed random variables with a common variance  $\sigma$ ,

$$P_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}; \quad Z = \sqrt{X^2 + Y^2} \leq z \Rightarrow \quad (C.3)$$

$$F_Z(z) = \text{Pr ob}\{(x, y): \sqrt{x^2 + y^2} \leq z\}; \quad F_Z \equiv \text{cumulative distribution function, Prob} \equiv \text{probability} \quad (C.4)$$

$$\begin{aligned} \text{Let } x &= r\cos\theta, y = r\sin\theta, r = \sqrt{x^2 + y^2} \\ \Rightarrow F_Z(z) &= \frac{1}{2\pi\sigma^2} \int_{x^2+y^2 < z^2} \int e^{-\frac{x^2+y^2}{2\sigma^2}} dx dy = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_0^z e^{-\frac{r^2}{2\sigma^2}} r dr d\theta \end{aligned} \quad (C.5)$$

Performing the double integration, the cumulative distribution function for the random variable  $Z$  is

$$F_Z(z) = -e^{-\frac{z^2}{2\sigma^2}} + 1 \Rightarrow P_Z(z) \equiv \frac{d}{dz}[F_Z(z)] = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}} \quad (C.6)$$

This is the formula for the Rayleigh probability density function, and therefore, the proof is complete.

**APPENDIX D**

**PROOF OF THE EQUIVALENCE OF A COMPLEX VALUED  
RANDOM VARIABLE TO A TWO-DIMENSIONAL RANDOM VECTOR**

The following is the proof of theorem 1, given in this report. Recall that this is the theorem to claim that a complex valued random variable is statistically equivalent to its random vector and bivariate ordered pair counterparts. The theorem is stated again here.

Theorem 1 (Repeat): Given a complex valued random variable,  $X+iY$ , there exists a two dimensional random vector, namely  $(X, Y)^T$ , (or bivariate distributed ordered pair, namely  $(X,Y)$ ) that is statistically equivalent to the complex valued random variable.

Proof: Suppose  $X$  and  $Y$  are random variables. Let  $V$  be a random vector such that  $V^T=(X,Y)$ , where the superscript  $T$  refers to the transpose of a vector (matrix). Then by the mathematical definition of a random vector,  $V$  is a function that can be written in the following form:

$$V: \Omega \rightarrow \mathbb{R}^2 \text{ such that } V(\omega) = \begin{bmatrix} S(\omega) \\ Q(\omega) \end{bmatrix} \forall \omega \in \Omega, \text{ a sample space} \quad (D.1)$$

Notice that there exists a one to one and onto mapping, namely  $g(x,y)=x+iy$ , that maps from the  $\mathbb{R}^2$  space to the complex plane,  $C$ . Combining this map with the random vector mapping gives the composition mapping,  $Z = g \circ V: \Omega \rightarrow \mathbb{R}^2 \rightarrow C$ . With this construction along with the mathematical definition of probability on a random variable, notice that

$$\begin{aligned} P[x + iy \in A \subseteq C] &= P[Z \in A \subseteq C] \equiv P\{\omega \in \Omega: Z(\omega) \in A \subseteq C\} \\ &= P\{\omega \in \Omega: V(\omega) \in g^{-1}(A) \subseteq \mathbb{R}^2\} \equiv P[(x, y) \in g^{-1}(A) \subseteq \mathbb{R}^2] \\ &= P\{(x, y): g(x, y) = x + iy\} \end{aligned} \quad (D.2)$$

Pictorially, this can be seen in the following way:

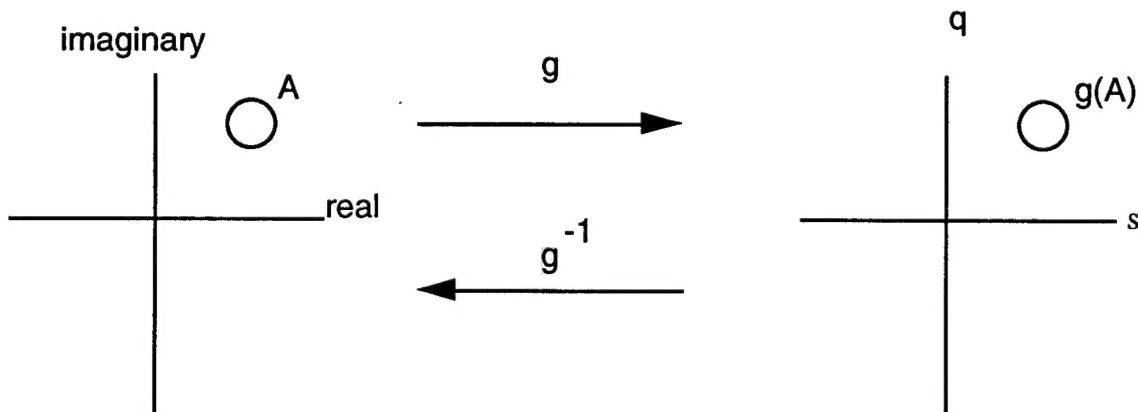


FIGURE D-1. ONE TO ONE ONTO TRANSFORMATION OF EVENT A

These arguments shows that probabilistically,  $Z=X+iY$  and  $V=(X,Y)$  must be equivalent, or analogously,  $Z=X+iY$  and the vector  $V^T=(X,Y)^T$  must be equivalent. Therefore, the ordered pair notation and complex notation of a pair of random variables are statistically equivalent. This completes the proof of the theorem.

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